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Anomalous roughness exponent of growing interfaces in a disordered medium

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We study the roughness properties of an interface that is driven through a random medium. The growth is modeled by a continuous stochastic equation with quenched noise. A new intermediate scaling regime is introduced and analyzed. Effective self-affine properties in this regime allow us to define an effective roughness exponent $\alpha_{eff}\simeq 0.8$ in excellent agreement with previous experiments.

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In recent years the motion of a nonequilibrium interface in a disordered environment has attracted much attention [1,2]. Applications range from wetting phenomena to motion of polymers through random media, however, the typical experimental realization of these phenomena is the fluid flow in a porous medium. Experiments have been performed [3–7] to characterize the immiscible displacement of a fluid by one that wets the medium more effectively. In this situation, the interface separating both fluids is rough and self-affine.

The effect of disorder on the morphology of growing interfaces is crucial. When the disorder is caused by thermal fluctuations, the random field changes both in space and time (e.g., ballistic deposition, aggregation, Eden model, and many other growth processes [1]) and the resulting dynamics are described by stochastic equations of the type introduced by Kardar, Parisi, and Zhang (KPZ) [8], which are well understood. On the contrary, if the random medium is frozen in time (e.g., a porous medium) a natural way to model its effects is by introducing a time-independent or quenched disorder, as has been done by some authors [10–17]. Less is known about these last models where the motion of the interface is dominated by the pinning forces present in the inhomogeneous medium.

In this Rapid Communication we study the quenched model at zero temperature [10–16] in which a d-dimensional interface, characterized by its height $h(\vec{x},t)$ at position $\vec{x}$ and time t, moves in a $(d+1)$-dimensional disordered medium. The interface grows obeying the stochastic differential equation:

$$\frac{\partial}{\partial t}h(\vec{x},t)=\nabla^2 h(\vec{x},t)+F+\eta(\vec{x},h), \tag{1}$$

where the surface tension effects are modeled by the diffusive term, F is an external driving force, and η is an uncorrelated random field with zero mean value. An important characteristic associated with this model is that a threshold phenomenon occurs because the above mentioned pinning forces are able to dramatically slow down the motion of the interface in large regions. Thus, an interface moves with a finite velocity $v(F)$ if the driving force exceeds a critical value F_c, and it is pinned by the disorder for $F<F_c$.

The interface width averaged over a region of linear dimension L,

$$\sigma(L,t)=\langle[h(\vec{x},t)-\langle h(\vec{x},t)\rangle]^2\rangle^{1/2}, \tag{2}$$

is a characterization of the statistical fluctuations and saturates at long times:

$$\sigma(L,t)\sim\begin{cases} t^{\beta} & \text{if} \quad t\gg t_s \\ L^{\alpha} & \text{if} \quad t\ll t_s, \end{cases} \tag{3}$$

where α is the roughness exponent and β is the time exponent. The saturation time, t_s, depends on the system size L. Here, we are interested in studying the behavior of the fluctuations near the pinning transition.

Another important quantity is the horizontal correlation length $l_c(t)\sim t^{1/z}$ where z is the so-called dynamical exponent. This correlation length is related to the saturation time because the saturation is reached when the condition $l_c(t)\sim L$ is satisfied. Above the critical point, the diffusion length is the only relevant scale and the correlation length behaves as $l_c(t)\sim t^{1/2}$. Only at the depinning threshold $(F=F_c)$ can the diffusive behavior change. In the limit of strong pushing $(F\gg F_c)$, the interface moves very fast and the quenched model reduces to the linear KPZ equation (Edwards-Wilkinson model [18]), for which the exponents are known: $\beta=1/4$, $\alpha=1/2$, and $z=2$ for $d=1$.

Recently, much analytical and numerical work has been carried out to understand the pinning transition and to obtain the critical exponents, but experiments, theory, and simulations have not led to a consistent picture yet (see [2] for recent reviews). Both renormalization group (RG) approximations [13] of (1) and Imry-Ma arguments [19] allow us to obtain the values for the roughness exponent ($\alpha=1$) and time exponent ($\beta=3/4$) for dimension $d=1$ in disagreement with a number of numerical models in 1+1 dimensions. Martys *et al.* [20] and Nolle *et al.* [9] in a wetting invasion model found $\alpha=0.8$ and the discretized version of Eq. (1) performed by Kessler *et al.* [12] gave $\alpha\simeq 0.76$. Directed percolation and solid-on-solid type models gave $\alpha\simeq 0.63$ [7,15]. Similarly, the roughness exponent measured in experiments range from Rubio *et al.* ($\alpha=0.73\pm 0.03$ [3]) or Horvath *et al.* ($\alpha=0.88\pm 0.08$ [5] and $\alpha\simeq 0.81$ [4]) to Buldyrev *et al.* ($\alpha\simeq 0.65$ [7]).

In an enlighting paper, Amaral *et al.* [21] have shown that the inclusion of the nonlinear term $\lambda(\nabla h)^2$ in Eq. (1) gives a roughness exponent $\alpha\simeq 0.63$, in agreement with the mapping to directed percolation but models in the universality class of Eq. (1) lead to a larger roughness exponent, $\alpha\simeq 0.75-0.88$. As has been shown by Tang *et al.* [22] these two different universality classes are related to isotropy properties of the random background. The model posed in Eq. (1) describes growing interfaces in an isotropic medium. On the contrary, the KPZ term $\lambda(\nabla h)^2$ can be present for interfaces in anisotropic random media [22]. Most solid-on-solid type models are expected to be in this category. Although this last work clarifies the situation, one question remains open: What is the reason for the difference between RG predictions and results from both experiments and simulations for the model (1)?

Our main purpose here is to show that there is a crossover regime in (1+1) dimensions for the model described by (1) with an effective roughness exponent $\alpha_{eff} \simeq 0.8$. As we shall see later, this intermediate regime, which appears for F somewhat larger than F_c, has new scaling properties. We speculate that this result may give an explanation of the low value for α measured experimentally.

First, we study the statistical geometry or topography on the random medium for $d=1$. For any F, let us consider the substrate divided into two types of regions: "pinning (or trapping) regions" and "pushing regions." The first one is formed by points (x,y) verifying that $F+\eta(x,y)<0$, consequently they tend to keep the interface pinned and the motion is slowed down inside the pinning regions. However, a piece of interface in a pushing region will be able to move fast, since $F+\eta(x,y)>0$ and there are no trapping sites. Two characteristic lengths corresponding to the mean size of these regions may be considered, ξ_+ and ξ_-, which are the characteristic size of a pushing and trapping region, respectively. At the pinning transition, with a pushing force $F=F_c$, the characteristic length of a pinning region is $\xi_- \sim L$, thus the interface remains pinned inside a pinning region, which occupies the substrate almost completely. By increasing F, ξ_- is reduced and it becomes zero rapidly, while the characteristic length of a pushing region is $\xi_+ \sim L$ now. This means that for $F \gg F_c$, ξ_+ is much larger than ξ_-, most of the interface is inside a pushing region and moves fast, so the long time limit corresponds to the Edwards-Wilkinson universality class. As usual, near the critical point, ξ_- is a relevant length scale of the problem with a typical scaling law $\xi_- \sim (F-F_c)^{-\nu}$ [23]. Numerical experiments [9], that are believed to be in the universality class described by Eq. (1), have obtained $\nu \simeq 4/3$ for the correlation length exponent.

In view of the previous arguments, we shall calculate the interface width at time t averaging over both pinning and pushing regions. Suppose that an interface is driven by a pushing force F and L is the system size, then from Eq. (2) we can write

$$\sigma^2(L,t) = (1/L)\int_0^L dx[h(x,t) - \langle h(x,t)\rangle]^2,$$

which can be evaluated over N_+ pushing segments and N_- pinning segments as

$$\sigma^2(L,t) = \frac{1}{L}[N_+\xi_+(L,F)\sigma_+^2(t) + N_-\xi_-(L,F)\sigma_-^2(t)],$$

where σ_+ and σ_- are the interface widths inside a pushing and pinning region, respectively. As local quantities these interface widths do not depend on the system size nor on the pushing force F; they only represent the typical behavior of the interface inside a pushing or trapping region and can be determined by physical arguments. Taking into account that $N_+\xi_+ + N_-\xi_- \simeq L$ the last equation is expressed as follows:

$$\sigma^2(L,t) \sim \left(1 - \frac{N_-\xi_-}{L}\right)\sigma_+^2(t) + \left(\frac{N_-\xi_-}{L}\right)\sigma_-^2(t). \quad (4)$$

Let us see how we can calculate $\sigma_\pm(t)$ with simple arguments already used in the literature [11,12,14]. Since the interface velocity vanishes for large times inside a pinning region, as we have discussed before, we can neglect the height dependence of the noise in Eq. (1) to describe the behavior of a piece of interface moving through this kind of region:

$$\frac{\partial}{\partial t}h(x,t) = \frac{\partial^2}{\partial x^2}h(x,t) + F + \eta(x),$$

which gives immediately $\sigma_-(t) \sim t^{3/4}$ [14]. In the same way, a piece of interface that moves inside a pushing region of the disordered medium has a high velocity $[v(F) \simeq F]$ and Eq. (1) becomes approximatively

$$\frac{\partial}{\partial t}h(x,t) = \frac{\partial^2}{\partial x^2}h(x,t) + F + \eta(x,Ft).$$

So, inside a pushing region the behavior of the interface reduces to the case of time-dependent noise and it is well known that $\sigma_+(t) \sim t^{1/4}$ [11,12].

A more reduced expression can be obtained when we analyze Eq. (4) in the regimes of interest. Letting $F \to F_c$ as we have discused above, only one pinning region appears in the random medium, thus $N_- \sim 1$ and the expression for the interface width, Eq. (4), is dominated by the second term, yielding

$$\sigma(L,t) \sim \left(\frac{\xi_-(L,F)}{L}\right)^{1/2} t^{3/4} \quad (5)$$

and in the opposite limit $F \gg F_c$, far from the transition, we have

$$\sigma(L,t) \sim \left(\frac{\xi_+(L,F)}{L}\right)^{1/2} t^{1/4}. \quad (6)$$

At this point, we would like to analyze in detail the two regimes given in Eqs. (5) and (6). First, in the strong pushing regime, the exponents can be calculated in a straightforward manner since a simple comparison between (6) and (3) leads to $\beta = 1/4$ in this regime. As usual, at times larger than the saturation time t_s, the width in Eq. (6) scales as $\sigma(L) \sim \xi_+^{1/2} L^{-1/2} t_s^{1/4}$, where t_s is given by the condition $t_s \sim L^2$ and as we have discussed in a previous paragraph the characteristic size of a pushing region is $\xi_+ \sim L$ in the strong pushing regime. Thus, we conclude that $\alpha = 1/2$ in this case. So, we have recovered the exponents of the Edwards-Wilkinson model, as expected.

A more careful analysis is needed in the intermediate regime given by Eq. (5) and this is the regime of interest for us. Mostly, the difficulties arise from the dependence on L of $\xi_-(L,F)$ which is unknown and can be complicated. Only a power law behavior $\xi_-(L,F) \sim L^\theta$ yields a roughness exponent α for times larger than the saturation time. To see that this power law is achieved we must turn to (1) and observe some symmetry properties in our model. For any factor λ, we can rescale (1) obtaining

$$\frac{\partial[\lambda h(x,t)]}{\partial(\lambda^2 t)} = \frac{\partial^2[\lambda h(x,t)]}{\partial(\lambda x)^2} + \frac{F}{\lambda} + \eta(\lambda x, \lambda h),$$

and the equation remains invariant [24]. In other words, multiplying F by some factor λ is equivalent to rescaling our variables $x'=\lambda x$, $h'=\lambda h$, and $t'=\lambda^2 t$ with fixed F. So, it is possible to rescale any distance depending on both system size and pushing force F as

$$\xi(L,\lambda F)=\lambda^{-1}\xi(\lambda L,F)$$

and in the same way for times

$$\tau(L,\lambda F)=\lambda^{-2}\tau(\lambda L,F).$$

This symmetry allows us to convert any scaling law with F to a scaling law with the system size L. In particular, the size of a pushing region which scales as $\xi_-(L,\lambda F)\sim(\lambda F-F_c)^{-\nu}$ can be written as $\xi(\lambda L,F)\sim\lambda^{1-\nu}$ and the power law behavior is satisfied for $\theta=1-\nu$. It is worth stressing here that this invariance is a consequence of the linear character of (1) and it would be broken down by adding a general nonlinear term (e.g., $[\nabla h(x,t)]^2$).

Now, we are able to calculate the exponents in the intermediate regime, in which (5) describes the interface width. An inspection of Eq. (5) immediately gives the temporal exponent $\beta=3/4$ in agreement with both previous simulations [14,16] and RG calculation close to the pinning threshold. Next, if we replace in (5) the behavior of $\xi_-(L,F)$ to varying L with fixed F, at saturation time we have

$$\sigma(L)\sim L^{(3-\nu)/2}, \tag{7}$$

which implies an effective roughness exponent $\alpha_{eff}=(3-\nu)/2$ in the intermediate regime. Thus, from the measured value $\nu\simeq 4/3$ we obtain an effective exponent $\alpha\simeq 0.8$ in good agreement with both experiments and simulations.

At long times a crossover to the Edwards-Wilkinson model must occur and the intermediate regime can be viewed like a crossover regime [25]. Surprisingly, an apparent roughness exponent may be defined in this crossover regime. Several effective models have been recently proposed in which an effective roughness exponent was found [16,26]. This exponent seems to be a very plausible explanation of the numerical results obtained until now.

In summary, the results presented here seem to imply that the anomalous roughness exponent observed in wetting invasion experiments is an effective exponent. An invariance in (1) leads to a crossover regime with effective scaling properties. This intermediate regime appears for F somewhat larger than F_c and we speculate that due to the anomalous scaling (7), apparent roughness exponents have been measured in both experiments and simulations.

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[24] This property was pointed out by Kessler, Levine, and Tu in Ref. [12] and used in Ref. [16] as well.
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